Cosmology on a Brane in Minkowski Bulk

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Abstract.

We discuss the cosmology of a 3-brane embedded in a 5D bulk space-time with a cosmological constant when an intrinsic curvature Ricci scalar is included in the brane action. After deriving the 'brane-Friedmann' equations for a Z_2 symmetrical metric, we focus on the case of a Minkowski bulk. We show that there exist two classes of solutions, close to the usual Friedmann-Lemaître-Robertson-Walker cosmology for small enough Hubble radii. When the Hubble radius gets larger one either has a transition to a fully 5D regime or to a self-inflationary solution which produces a late accelerated expansion. We also compare our results with a perturbative approach and eventually discuss the embedding of the brane into the Minkowski space-time. This latter part of our discussion also applies when no intrinsic curvature term is included.

1 Introduction

A lot of interest has recently been raised for field theories where the standard model of high energy physics is assumed to live on a surface (called generically a brane) embedded in a larger space-time. The (super)gravitational fields are in contrast usually considered to live in the whole space-time. Models coming from string-M theory like the Hořava-Witten walls [1], or D-branes, as well as

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from a more phenomenological approach [2, 3] have been extensively studied in particular in cosmology.

A question, which naturally arises, is how to recover standard gravity in its well tested perturbative regime. A first approach is to assume that the dimensions transverse to the brane are compact, in which case the usual Kaluza-Klein results allow to recover 4D gravity on scales which are larger than the size of the extra dimensions. The main difference with the old picture being there that, because the standard model fields are assumed to be brane-localized, one can have very large extra-dimensions in comparison to the length scales probed by high energy physics [2]. On scales smaller than the size of the extra dimensions, on the other hand, gravity enters a higher dimensional regime. Another way of recovering usual 4D gravity on the brane for large distances is to embed a positive tension 3-brane into an AdS_5 bulk [3] in which case the crossover scale between 4D and 5D gravity is set by the AdS radius. In this latter case, the extra dimension has also a finite size.

We will be mainly here interested in a more recent approach advocated by Dvali et al. [4, 5]. In this approach, the 3-brane is embedded in a space-time with infinite size extra dimensions, with the hope that this picture could shed new light on the standing problem of the cosmological constant as well as on supersymmetry breaking [4, 6]. The recovery of the usual gravitational laws is obtained by adding to the action of the brane an Einstein-Hilbert term computed with the brane intrinsic curvature. The presence of such a term in the action is generically induced by quantum corrections coming from the bulk gravity and its coupling with matter living on the brane and should be included for a large class of theories for self-consistency (see e.g. [5, 7]). In the particular case of a 3-brane embedded in a 5D Minkowski space-time, Dvali, Gabadadze and Porrati have shown that one recovers a standard 4D Newtonian potential for small distances, whereas gravity is in a 5D regime for large distances. The tensorial structure of the graviton propagator in this theory has in contrast been shown to be higher dimensional which is likely to rule out the theory from a phenomenological point of view. For a brane embedded in a bulk with 2 or more extra dimensions, one can show that the theory is always 4-dimensional [5] for a zero thickness brane ².

Our purpose is here to study the cosmology of these models in the case of a 5D bulk. Although, as mentionned above, such a theory has serious phenomenological problems, its cosmology can help to have a better understanding of this kind of model and of the idea of gravity localization through an intrinsic curvature term on the brane. In particular, exact cosmological solutions, as given in this work, provide a unique way to test the theory in its fully non linear regime. The Friedmann-like equations governing the cosmological evolution of a brane possessing an intrinsic curvature term in its action have already been derived and

²however, a simple regularization scheme in the graviton propagator accounting for a brane non zero thickness leads to a behaviour similar to the 5D case [8].

discussed for an AdS-Schwarzschild bulk space-time [9, 10, 11]. In the first part of this paper we derive similar equations (valid whenever the bulk matter is a pure cosmological constant) in a slightly different way, following the work of Binétruy et al. [12, 13]. For this pupose we will adopt a brane-based coordinate system and specialize to a Z_2 symmetrical metric. We then discuss these equations for a vanishing bulk cosmological constant as well as for a vanishing Schwarzschild mass parameter. We show that there exist two possible types of cosmology which are both similar to the usual 4D Friedmann-Lemaître-Robertson-Walker (FLRW) cosmology when the Hubble radius is small. We point out, however, a discrepancy between the Newton's constant inferred by cosmology in that regime, and the one defined by a Cavendish-like experiment. For larger Hubble radius, the cosmology either evolves to the brane-typical linear relationship between the Hubble parameter and energy density, or to a brane self-inflationary solution previously noticed by Shtanov [10]. In this latter case, one has a late-time accelerated expansion sourced by the intrinsic curvature term of the brane itself (and not by its tension). In the last part of this paper we interpret the found two-folded cosmology by looking at the brane embedding into the bulk space-time. We give in particular the change of coordinates between our brane-based metric and a canonical Minkowskian bulk metric.

2 4D gravity on a 3-brane in 5D Minkowski bulk

We consider a 3-brane embedded in a 5D space-time with an intrinsic curvature term included in the brane action. We can take accordingly the following action

$$S_{(5)} = -\frac{1}{2\kappa^2} \int d^5 X \sqrt{-\tilde{g}} \tilde{R} + \int d^5 X \sqrt{-\tilde{g}} \mathcal{L}_m, \tag{1}$$

to which we add the brane curvature term

$$-\frac{1}{2\mu^2} \int d^4x \sqrt{-g} R. \tag{2}$$

The first term in (1) corresponds to the Einstein-Hilbert action in five dimensions for a five-dimensional metric \tilde{g}_{AB} (bulk metric) of Ricci scalar \tilde{R} . Similarly, (2) is the Einstein-Hilbert action for the induced metric g_{cd} on the brane, R being its scalar curvature. The induced metric³ g_{cd} is defined as usual from the bulk metric \tilde{g}_{AB} by

$$g_{cd} = \partial_c X^A \partial_d X^B \tilde{g}_{AB}, \tag{3}$$

³Throughout this article, we will adopt the following convention for indices: upper case Latin letters A, B, ... will denote 5D indices: 0, 1, 2, 3, 5; lower case Latin letters from the beginning of the alphabet: c, d, ..., will denote 4D indices parallel to the brane: 0, 1, 2, 3; lower case Latin letters from the middle of the alphabet: i, j, ..., will denote space-like 3D indices parallel to the brane: 1, 2, 3.

where $X^A(x^c)$ represents the coordinates of an event on the brane labelled by x^c . We point out that the Einstein-Hilbert term (2), if not present at classical level, should appear quite generically (namely for any brane with a non conformal world volume theory [5]) from quantum corrections so that its inclusion is dictated by self consistency. The second term in (1) corresponds to the 'matter' content. Aside from bulk matter, we have included there the contribution of the brane-localized matter. This latter contribution can be further rewritten

$$\int d^4x \sqrt{-g} \left(\lambda_{brane} + l_m\right),\tag{4}$$

where l_m represents the lagrangian density of 'matter' fields living on the brane, and λ_{brane} the brane tension (or 'cosmological constant'). We stress here that this tension is unrelated to the presence of the intrinsic curvature term (2), and can in principle be tuned to be zero [4, 5]. We also define from the dimensionful constants κ and μ the related quantities

$$\kappa^2 = 8\pi G_{(5)} = M_{(5)}^{-3},\tag{5}$$

and

$$\mu^2 = 8\pi G_{(4)} = M_{(4)}^{-2}. (6)$$

For a brane embedded in a Minkowski space-time with an action obtained by adding (1) and (2), Dvali *et al.* [4] have shown that the usual 4D Newton's law, for static point like sources on the brane, is recovered at small distances on the brane; whereas at large distances the gravitational force is given by the 5D $1/r^3$ law. The crossover length scale between the two different regimes is given by

$$r_0 = \frac{M_{(4)}^2}{2M_{(5)}^3}. (7)$$

However, the model is plagued by the presence of an extra scalar polarization degree of freedom which can be seen by looking at the tensor structure of the graviton propagator. This structure (analogous to the one of a 5D massless graviton) is also responsible for a rescaling of the Newton's constant with respect to the usual 4D one. Namely, if one wants to define the Newton's constant through some Cavendish like experiment, looking at the force exerted between static point-like particles in the regime where this force obey the usual 4D Newton's law (i.e. for distances smaller than r_0), then one finds the following relationship [14] between the so-defined Newton's constant G_N and $G_{(4)}$:

$$G_N = \frac{4}{3} \frac{\mu^2}{8\pi} = \frac{4}{3} G_{(4)}.$$
 (8)

This rescaling is due to the presence of the extra scalar degree of freedom which exerts an extra attraction with respect the the ordinary case, it does not persist for more than one extra dimension and a zero thickness brane [5].

3 Brane Friedmann equations

The purpose of this section is to derive Friedmann like equations for the brane metric. We will consider five-dimensional spacetime metrics of the form

$$ds^2 = \tilde{g}_{AB}dx^A dx^B = g_{cd}dx^c dx^d + b^2 dy^2, \tag{9}$$

where y is the coordinate of the fifth dimension and we will adopt a brane-based approach where the brane is the hypersurface defined by y = 0. Being interested in cosmological solutions, we take a metric of the form

$$ds^{2} = -n^{2}(\tau, y)d\tau^{2} + a^{2}(\tau, y)\gamma_{ij}dx^{i}dx^{j} + b^{2}(\tau, y)dy^{2},$$
(10)

where γ_{ij} is a maximally symmetric 3-dimensional metric (k = -1, 0, 1 will parametrize the spatial curvature).

The five-dimensional Einstein equations take the form

$$\tilde{G}_{AB} \equiv \tilde{R}_{AB} - \frac{1}{2}\tilde{R}\tilde{g}_{AB} = \kappa^2 \tilde{S}_{AB},\tag{11}$$

where \tilde{R}_{AB} is the five-dimensional Ricci tensor, and the tensor \tilde{S} is the sum of the energy momentum tensor \tilde{T} of matter and the contribution coming from the scalar curvature of the brane. We denote this latter contribution \tilde{U} . We have

$$\tilde{S}^{A}_{B} = \tilde{T}^{A}_{B} + \tilde{U}^{A}_{B},$$
 (12)

where the energy momentum tensor can be further decomposed into two parts

$$\tilde{T}^{A}{}_{B} = \check{T}^{A}{}_{B}|_{\text{bulk}} + T^{A}{}_{B}|_{\text{brane}}.$$

$$\tag{13}$$

The first tensor $\check{T}^A{}_B|_{\text{bulk}}$ is the energy momentum tensor of the bulk matter, which will be assumed in the present work to be that of a cosmological constant:

$$\check{T}^{A}_{B}|_{\text{bulk}} = \operatorname{diag}\left(-\rho_{B}, -\rho_{B}, -\rho_{B}, -\rho_{B}, -\rho_{B}\right),$$
(14)

where the energy density ρ_B is a constant. The second term $T^A_{B|_{\text{brane}}}$ in (13) corresponds to the matter content in the brane (y=0). Since we consider here only strictly homogeneous and isotropic geometries inside the brane, the latter can be expressed quite generally in the form

$$T^{A}_{B}|_{\text{brane}} = \frac{\delta(y)}{b} \operatorname{diag}(-\rho_{b}, p_{b}, p_{b}, p_{b}, 0), \qquad (15)$$

where the energy density ρ_b and pressure p_b are independent of the position inside the brane, i.e. are functions only of time. The possible contribution of a non zero λ_{brane} (defined in (4)) will be assumed in the following to be included in ρ_b and p_b . We also assume that there is no flow of matter along the fifth dimension, which gets translated to

$$\tilde{T}_{05} = 0.$$
 (16)

The non vanishing component of \tilde{U} are straightforwardly given by

$$\tilde{U}_{00} = -\frac{3\delta(y)}{\mu^2 b} \left\{ \frac{\dot{a}^2}{a^2} + k \frac{n^2}{a^2} \right\}, \tag{17}$$

$$\tilde{U}_{ij} = -\frac{\delta(y)}{\mu^2 b} \left\{ \frac{a^2}{n^2} \gamma_{ij} \left(-\frac{\dot{a}^2}{a^2} + 2\frac{\dot{a}}{a}\frac{\dot{n}}{n} - 2\frac{\ddot{a}}{a} \right) - k\gamma_{ij} \right\},\tag{18}$$

where a dot stands for a derivative with respect to τ . Our aim is now to obtain the equations governing the cosmological evolution of the induced metric on the brane.

Let us first remind the results obtained by Binétruy et al. [13] which we will use in the following. For the particular setting described above, one can obtain a first integral of the Einstein's equations (11) in the bulk which can be written

$$\frac{(a'a)^2}{b^2} - \frac{(\dot{a}a)^2}{n^2} - ka^2 + \frac{\kappa^2}{6}a^4\rho_B + \mathcal{C} = 0,$$
 (19)

where C is a constant of integration. In the above equation, a prime stands for a derivative with respect to y. One can show that any set of functions $\{a, b, n\}$ satisfying (19) together with $\tilde{G}_{05} = 0$ (which is here mandatory from (16)) will be solution of whole the Einstein's equations in the bulk. The brane is then taken into account by using Israel's junction conditions [15]. These junction conditions relate the jump across the brane of the brane extrinsic curvature to the delta functions sources on the right hand side of Einstein's equations (11). The extrinsic curvature tensor of a given y = constant surfaces in the background metric (10) is given by

$$K_B^A = \operatorname{diag}\left(\frac{n'}{nb}, \frac{a'}{ab}\delta_j^i, 0\right). \tag{20}$$

We find here for the junction conditions (see e.g. [12, 13] for more details)

$$\frac{[a']}{a_0 b_0} = -\frac{\kappa^2}{3} \rho_b + \frac{\kappa^2}{\mu^2 n_0^2} \left\{ \frac{\dot{a}_0^2}{a_0^2} + k \frac{n_0^2}{a_0^2} \right\}, \tag{21}$$

$$\frac{[n']}{n_0 b_0} = \frac{\kappa^2}{3} (3p_b + 2\rho_b) + \frac{\kappa^2}{\mu^2 n_0^2} \left\{ -\frac{\dot{a}_0^2}{a_0^2} - 2\frac{\dot{a}_0 \dot{n}_0}{a_0 n_0} + 2\frac{\ddot{a}_0}{a_0} - k\frac{n_0^2}{a_0^2} \right\}$$
(22)

where the subscript 0 for a, b, n means that these functions are taken in y = 0, and $[Q] = Q(0^+) - Q(0^-)$ denotes the jump of the function Q across y = 0. We can compare (21) and (22) with the similar equations obtained when we consider the model given only by action (1). These equations can be recovered from (21) and (22) by letting μ go to infinity. One sees then that the brane intrinsic curvature

term (2) acts as a source for the Einstein's equations which depends explicitly on the induced metric time derivatives as well as on its spatial curvature. It is also apparent from (12), (17) and (18). For a given induced metric parametrized by a_0 , n_0 and k, the intrinsic curvature term (2) acts as a 'cosmic fluid' of density ρ_{curv} and pressure p_{curv} given by

$$\rho_{curv} = -\frac{3}{\mu^2 n_0^2} \left\{ \frac{\dot{a}_0^2}{a_0^2} + k \frac{n_0^2}{a_0^2} \right\}$$
 (23)

$$p_{curv} = \frac{1}{\mu^2 n_0^2} \left\{ \frac{\dot{a}_0^2}{a_0^2} - 2 \frac{\dot{a}_0 \dot{n}_0}{a_0 n_0} + 2 \frac{\ddot{a}_0}{a_0} + k \frac{n_0^2}{a_0^2} \right\}$$
(24)

One note that the energy density of this 'fluid' is always negative whenever k = 0 or k = 1.

Assuming the symmetry $y \leftrightarrow -y$ for simplicity, the junction condition (21) can be used to compute a' on the two sides of the brane. We have in this case $[a'] = 2a'(0^+)$. By continuity when $y \to 0$, (19) yields the generalized (first) Friedmann equation:

$$\epsilon \sqrt{H^2 - \frac{\kappa^2}{6}\rho_B - \frac{\mathcal{C}}{a_0^4} + \frac{k}{a_0^2}} = \frac{\kappa^2}{2\mu^2} \left(H^2 + \frac{k}{a_0^2} \right) - \frac{\kappa^2}{6}\rho_b, \tag{25}$$

where the Hubble parameter H is defined here by

$$H = \frac{\dot{a}_0}{a_0 n_0},\tag{26}$$

and $\epsilon = \pm 1$ is the sign of [a'] (see equation (21)). A similar equation has been obtained in [10] for an AdS-Schwarzschild background (in which case ρ_B is negative, see also [9, 11]) but only the case $\epsilon = -1$ was discussed there. As we will see more explicitly in section (4.3) the two different possible ϵ correspond to two different embeddings of the brane into the bulk space-time (see also [16, 17]). Before specializing to the case where ρ_B is set to zero, we further note that one can derive for the brane matter a usual conservation equation. If we plug into the (0,5) component of the Einstein's equations the jump conditions (21) and (22) we obtain, as when no brane intrinsic curvature (2) is included, the conservation equation

$$\dot{\rho_b} + 3(p_b + \rho_b) \frac{\dot{a_0}}{a_0} = 0. \tag{27}$$

Equation (27) and (25), together with the brane matter equation of state, are then sufficient to derive the cosmological evolution of the brane metric. Last, we recall the brane-Friedmann equation obtained [13, 18, 19] when no curvature term (2) is included. It can easily be deduced from (25) by letting μ go to

⁴The 4D Bianchi indentities $\nabla^c G_{cd} = 0$ ensure that the energy-momentum of this 'fluid' is conserved.

infinity and reads (we write it here for a zero bulk cosmological constant and a zero Schwarzschild mass parameter C)

$$H^2 + \frac{k}{a_0^2} = \frac{\kappa^4}{36} \rho_b^2. \tag{28}$$

We will refer to a regime where such an equation is (approximately) valid as a fully 5D regime.

4 Discussion

We want now to discuss the solutions to the Friedmann-like equation (25) together with (27) when the bulk cosmological constant ρ_B vanishes. We recall that one can interpret the constant \mathcal{C} appearing in (25) as coming from the 5 dimensional bulk Weyl tensor [18, 20], since we are mainly interested here in Minkowskian bulk (for which the Weyl tensor vanishes) we will also set \mathcal{C} to zero in the following discussion.

4.1 Recovery of standard cosmology

Equation (25) can straightforwardly be rewritten

$$\frac{\mu^2}{3}\rho_b = H^2 + \frac{k}{a_0^2} - 2\epsilon \frac{\mu^2}{\kappa^2} \sqrt{H^2 + \frac{k}{a_0^2}}.$$
 (29)

It is now apparent on (29) that the standard cosmology, namely the usual 4D Friedmann equation:

$$\frac{8\pi G_{(4)}}{3}\rho_b = H^2 + \frac{k}{a_0^2},\tag{30}$$

is recovered whenever the last term on the right hand side of (29) is subdominant with respect to the first two terms, namely when

$$\sqrt{H^2 + \frac{k}{a_0^2}} \gg 2\frac{\mu^2}{\kappa^2},$$
 (31)

or in terms of the Hubble radius H^{-1} , when (neglecting the spatial curvature term)

$$H^{-1} \ll \frac{M_{(4)}^2}{2M_{(5)}^3}$$
 (32)

This matches the scale r_0 (7) found by Dvali *et al.* setting the crossover between the 4D gravity and the 5D gravity regimes. We will however show in the next subsection that for larger Hubble radii, one does not necessarily enter into a fully 5D regime (28). We want here to emphasize that, even if the evolution of the universe when (32) is verified is close to the standard one, there is a mismatch between the 'Newton's constant' $G_{(4)}$ derived from the Friedmann equation (29), and simply given in terms of μ^2 by the usual relation (6), and G_N as defined in (8). This mismatch can be tracebacked to the presence of the extra scalar degree of freedom. We will come back to this question in the last section.

4.2 Late time cosmology

For Hubble radius for which (32) is not satisfied, there are two distinct behaviours depending on ϵ . Equation (29) can indeed be rewritten (assuming $\rho_b \geq 0$)

$$\sqrt{H^2 + \frac{k}{a_0^2}} = \epsilon \frac{\mu^2}{\kappa^2} + \sqrt{\frac{\mu^2}{3}\rho_b + \frac{\mu^4}{\kappa^4}}$$
 (33)

Let us start by examining the case where $\epsilon = -1$. It is easy to show, integrating the above equation, that for k = 0 or k = -1, and assuming for simplicity⁵ some usual equation of state for the brane matter of the form

$$p_b = w\rho_b \qquad \text{(with } w \ge -1),\tag{34}$$

that a_0 diverges for late time, so that the density of any matter (with w > -1) goes to zero for late time, and reaches a regime where it is small in comparison with μ^2/κ^4 (which is equivalent to saying that $\sqrt{H^2 + k/a_0^2} \ll r_0^{-1}$) ⁶. One can then expand (29) to obtain

$$\sqrt{H^2 + \frac{k}{a_0^2}} = \frac{1}{6}\kappa^2 \rho_b,\tag{35}$$

which is the fully 5D regime (28). One has thus a transition from a 4D regime to a 5D regime.

If $\epsilon = 1$, however, $\sqrt{H^2 + \frac{k}{a_0^2}}$ is always larger than H_{self} given by

$$H_{self} = \frac{2\mu^2}{\kappa^2}. (36)$$

And the expansion will never enter into a fully 5D regime. If we start from initial conditions verifying (32) (and again if k = 0 or k = -1), it is easy to show, since H is bounded from below by H_{self} that the scale factor a_0 goes to infinity at large time, which implies again that the brane matter energy density goes to

⁵the following discussion is also valid in a more general case where the brane energy density is the sum of that of different kinds of cosmic 'fluids': matter, radiation, cosmological constant, as long as this energy density reaches a regime where it is small in comparison with μ^2/κ^4 .

⁶when k=1, it is possible that the universe turns over before reaching the regime where $\rho_b \ll \mu^2/\kappa^4$.

zero for any matter verifying (34) with w > -1. In this quite generic case, one has a transition between a usual 4D FLRW cosmology governed by equation (30) and an inflationary solution where H is (approximately) constant and given by H_{self} . The $\epsilon = 1$ case is thus able to give a quintessence-like scenario where a phase of matter or radiation dominated cosmology is followed by a late phase of accelerated expansion.

Shtanov [10] (see also [11]), already noticed that setting ρ_b to zero in the brane Friedmann equations, one obtains, aside from the trivial $\dot{a}=0$ solution, a de Sitter like expansion, the self-inflationary solution quoted above. This last solution can easily be understood if we look at the 'cosmic fluid' defined by (23) and (24) in the case where we set H to a constant value. We have then, neglecting the spatial 3D curvature

$$\rho_{curv} = -p_{curv} = -\frac{3H^2}{\mu^2} = \text{constant}, \tag{37}$$

so that the intrinsic curvature term (2) acts as a negative cosmological constant on the brane. Since the sources on the brane enter quadratically the Friedmannlike equation (25) it can however lead to a de Sitter expansion governed by the brane Friedmann equation (28)

$$H^2 = \frac{\kappa^4}{36} \rho_{curv}^2,\tag{38}$$

which leads in turn to a given value for H, namely H_{self} . For this particular H, the induced metric is thus identical to the one, one would obtain from a brane without a self curvature term (2) but with a positive brane tension ρ_b given by

$$\rho_b = -\rho_{curv}. (39)$$

This last solution will be called in the following the TI (tension-inflationary) solution, whereas the self-inflationary solution will be referred to as the SI solution. Let us further stress here that the sign of $-\epsilon$ indeed measures the sign of the effective energy density on the brane as seen from the 5D bulk theory. This effective energy density can be defined from (21) and (23) as the sum of the brane matter energy density ρ_b and the curvature energy density ρ_{curv} . When ϵ is equal to 1, this effective energy density is negative (even if ρ_b is positive), so that it is possible that the $\epsilon = 1$ scenario can be plagued by some instability associated with a violation of a positive energy condition.

4.3 Brane embedding in Minkowski space-time

We want here to discuss briefly the embedding of the cosmological solutions into Minkowski space-time. We follow again [13] to obtain the bulk metric under the assumption that b does not depend on time (so that with an appropriate

redefinition of y, one can assume that b=1). Having again (16), the (0,5) component of Einstein's equations lead to

$$\frac{\dot{a}}{n} = \alpha(\tau),\tag{40}$$

where α is a function that depends only on time (and not on y). By a suitable change of time one can choose $n_0 = 1$ so that α is simply given by \dot{a}_0 . On the other hand, The (0,0) component of Einstein's equations can be straightforwardly integrated in the bulk to lead to ⁷

$$a^{2} = (\dot{a}_{0}^{2} + k) y^{2} + a_{0}[a']|y| + a_{0}^{2}.$$

$$(41)$$

which can in turn be expressed in terms of the induced metric and the energy-momentum tensor on the brane through the jump condition (21). The scale factor a is thus given by

$$a = a_0 \left\{ 1 + |y| \left(-\frac{\kappa^2}{3} \rho_b + \frac{\kappa^2}{\mu^2} \left(H^2 + \frac{k}{a_0^2} \right) \right) + y^2 \left(H^2 + \frac{k}{a_0^2} \right) \right\}^{1/2}, \tag{42}$$

and n can be obtained by (40). The unknown functions a_0 and ρ_b in (42) (as well as in the expression of n) should then be computed through equations (25) and (27) as well as by the use of the equation of state for the brane cosmic fluid. Considering again the $\mathcal{C} = 0$ case, and using (25), the bulk metric is given by

$$a_{\epsilon} = a_0 + \epsilon |y| \left(\dot{a}_0^2 + k \right)^{1/2},$$

$$n_{\epsilon} = 1 + \epsilon |y| \ddot{a}_0 \left(\dot{a}_0^2 + k \right)^{-1/2},$$

$$b_{\epsilon} = 1.$$

$$(43)$$

This space-time can be shown to be a slice of Minkowski space-time. Indeed in the particular case of k=0 and $\epsilon=-1$, Deruelle and Dolezel [21] obtained an explicit change of coordinate $Y^A=Y^A(X^A)$ to go to the canonical 5D Minkowskian metric

$$ds^{2} = -(dY^{0})^{2} + (dY^{1})^{2} + (dY^{2})^{2} + (dY^{3})^{2} + (dY^{5})^{2}.$$
 (44)

One can check that this change of coordinate is also valid for $\epsilon = +1$, it is given by

$$Y^{0} = a_{\epsilon} \left(\frac{r^{2}}{4} + 1 - \frac{1}{4\dot{a}_{0}^{2}} \right) - \frac{1}{2} \int \frac{a_{0}^{2}}{\dot{a}_{0}^{3}} \partial_{\tau} \left(\frac{\dot{a}_{0}}{a_{0}} \right) d\tau,$$

$$Y^{i} = a_{\epsilon} x^{i},$$

$$Y^{5} = a_{\epsilon} \left(\frac{r^{2}}{4} - 1 - \frac{1}{4\dot{a}_{0}^{2}} \right) - \frac{1}{2} \int \frac{a_{0}^{2}}{\dot{a}_{0}^{3}} \partial_{\tau} \left(\frac{\dot{a}_{0}}{a_{0}} \right) d\tau,$$

$$(45)$$

⁷when $\rho_B = 0$ and assuming again $y \leftrightarrow -y$ symmetry. We refer the interested reader to [13] for more details.

where $r^2 = x^i x^j \eta_{ij}$, and η_{ij} is here a flat euclidian 3D metric. A related work is the one of Mukhoyama et al. [20] which gives the change of coordinates from a static Schwarzschild-AdS bulk metric [22] to a gaussian normal system as we used in this work (see also [17]). For the metric (43) with $k \neq 0$, one can find as well a change of coordinate $Y^A = Y^A(X^A)$ leading to the canonical Minkowski metric (44). It can be defined for k = 1 by

$$Y^{0} = a_{\epsilon}\tilde{y} + \tilde{z},$$

$$Y^{i} = a_{\epsilon}\tilde{Y}^{i},$$

$$Y^{5} = a_{\epsilon}\tilde{Y}^{5},$$

$$(46)$$

where \tilde{Y}^i and \tilde{Y}^5 are functions of the x^i only and verify

$$(\tilde{Y}^5)^2 + \sum_{i=1,3} (\tilde{Y}^i)^2 = 1, \tag{47}$$

which defines a 3D k=1 maximally symetric space, and \tilde{y} and \tilde{z} are given by

$$\tilde{y} = \frac{\dot{a}_0}{\sqrt{\dot{a}_0^2 + k}},\tag{48}$$

$$\tilde{z} = k \int \frac{1}{\dot{a}_0} \partial_{\tau} \left(\frac{a_0}{\sqrt{\dot{a}_0^2 + k}} \right) d\tau. \tag{49}$$

Similarly for k = -1, we find

$$Y^{0} = a_{\epsilon} \tilde{Y}^{0},$$

$$Y^{i} = a_{\epsilon} \tilde{Y}^{i},$$

$$Y^{5} = a_{\epsilon} \tilde{y} + \tilde{z},$$

$$(50)$$

where \tilde{Y}^i and \tilde{Y}^0 are function of the x^i only and verify

$$-(\tilde{Y}^0)^2 + \sum_{i=1,3} (\tilde{Y}^i)^2 = -1, \tag{51}$$

which defines a 3D k = -1 maximally symetric space, \tilde{y} and \tilde{z} are still given by (48) and (49). The change of coordinates (45), (46) and (50) are of the form

$$Y^{A} = a_{\epsilon} \left(\tilde{Y}^{A} + \tilde{y}^{A} \right) + \tilde{z}^{A}, \tag{52}$$

with \tilde{Y}^A being a function of the x^i only, \tilde{y}^A and \tilde{z}^A are functions of time and do not depend on ϵ . One can then invert these change of coordinate to obtain

$$\epsilon |y| = f(Y^A), \tag{53}$$

with $f(Y^A)$ a function of the Y^A independent of ϵ . The brane is then the hypersurface \mathcal{H} (or a slice of it, see below) in Minkowski space-time defined by $f(Y^A) = 0$. We see clearly on (53) that for a given ϵ (see [16, 21]) the coordinate system y, τ, x^i provides a double covering of only one side of \mathcal{H} . The two copies of the side of \mathcal{H} which is kept are respectively given by the y > 0 and y < 0 half-spaces (in the coordinates X^A) and are glued together along the brane; this provides in turn a jump of the extrinsic curvature tensor across the brane. Moreover, for a given function a_0 , the $\epsilon = \pm 1$ choice corresponds to keeping one side or the other of \mathcal{H} . It is also clear from (53), as well as from the definition of ϵ , that the jump in the extrinsic curvature for the $\epsilon = \pm 1$ solutions and a given function a_0 are opposite to each other.

One can check for example that when

$$\frac{\dot{a}_0^2}{a_0^2} + \frac{k}{a_0^2} = H_0^2,\tag{54}$$

with H_0 being a constant and k=1, the flat coordinates Y^A verify

$$(Y^5)^2 + (Y^3)^2 + (Y^2)^2 + (Y^1)^2 - (Y^0)^2 = \frac{(1 + \epsilon H_0|y|)^2}{H_0^2}.$$
 (55)

We recover in this case the standard embedding of de Sitter space-time (here, the brane) into a 5D Minkowski space-time as a hyperboloid⁸ \mathcal{H}_0 defined by setting y to zero in equation (55). For the inflationary solution (54) with k=0 (and still H_0 a constant) one find easily that (55) is still verified with the further restriction that $Y^0(y=0,\tau) - Y^5(y=0,\tau) > 0$ (as can be seen from (45)). We recover the well known result (see e.g. [23]) that this inflationary solution only covers half of de Sitter space-time. For $\epsilon = -1$ (and still k=0), this is also in agreement with the interpretation of the Z_2 symmetric domain-wall space-time of refs. [24] as the interior of \mathcal{H}_0 [16]. This latter case corresponds with the TI solution, while the SI solution covers the exterior of \mathcal{H}_0 .

Eventually, we note that the dependence of the metric (43) on a_0 is the same as when no brane intrinsic curvature term (2) is included. So that the above discussion also applies in this case. The only difference between the two cases (with and without a brane intrinsic curvature term (2)) is the dynamics of a_0 , that is to say the brane trajectory into the bulk space-time (see [17] for a very complete study of this question).

5 Conclusions

Let us first summarize some of our main results. We have studied the cosmology of a \mathbb{Z}_2 symmetrical 3-brane embedded in a 5D Minkowskian space-time, when

⁸one can indeed verify from (46) that the brane covers the whole hyperboloid.

an intrinsic curvature term is added on the brane. We have shown in particular that the usual cosmology is recovered for Hubble radii smaller than the crossover scale between 4D and 5D gravity found by Dvali et al. [4] and given by $r_0 = M_{(4)}^2/2M_{(5)}^3$. If we consider a matter content (radiation, matter,...) such that the energy density is decreasing as the scale factor increases (and for k=0 or k=-1), then one has two possibilities depending on the initial conditions. The universe either evolves toward a fully 5D regime where the relation between the Hubble parameter and the energy density is linear, or it evolves toward a self-inflationary solution, where the inflation is sourced by the scalar curvature term on the brane. We have also shown that the Newton's constant which enters the Friedmann-like equation during the early standard like evolution of the universe differs from the one defined from the measurement of the gravitational force between static point sources.

Although this work was mainly intended in an effort to better understand the models with brane and intrinsic curvature term on the brane, and could shed some light on the most interesting cases where there is more than one extra dimension, one could also look at it in a more phenomenological perspective. If one wishes to have an acceptable model from a phenomenological point of view, one should first take care of the presence of the extra scalar degree of freedom in the graviton propagator with respect to standard 4D gravity. As far as cosmology is concerned, this extra degree of freedom is responsible for the difference between the 'Cavendish' Newton's constant G_N and the 'cosmological' Newton's constant $G_{(4)}$. As a consequence of that, if we set G_N to its measured value, $G_{(4)}$ will be smaller by a factor 3/4 than the usual Newton's constant entering into the usual Friedmann equations. This is for exemple responsible for a change in the rate of expansion during nucleosynthesis and may be marginally compatible we the actual bounds. However, this extra scalar degree of freedom can not be compatible with relativistic tests of General Relativity [4], and one should first of all be able to cancel its effect. Although some proposal exist [4], like adding an extra vector field in the model, this seems very hard to do in a realistic way. Another crucial question, as acknowledged by Dvali et al., is whether the crossover scale r_0 can be made large enough⁹. Setting $M_{(5)}$ to TeV (and knowing $M_{(4)}$) gives a scale [4] that is much too low, so that one needs a very small 5D Planck scale which seems very difficult to cook. After this pessimistic account, we would like to underline a possible virtue of the hereabove described solutions, namely the fact that the $\epsilon = 1$ scenario leads to an interesting alternative to produce a phase of late accelerated expansion as indicated by the SN data [25]. We note moreover, that this last scenario 'explains' in a natural way the right order of magnitude for the crossover scale (in terms of the Hubble radius) between

 $^{^{9}}$ Indeed the present work enables to put constraints on r_{0} from cosmology in order for the transition to the fully 5D regime or to the self-inflationary solution to happen at late enough time.

matter dominated cosmology and late accelerated expansion: this scale is set by r_0 which is also the crossover scale between 4D and 5D gravity and has thus to be of cosmological size (see e.g. [4, 26]). It would be interested to know if such features persist in a more acceptable model with one extra dimension or for more than one transverse extra dimension.

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